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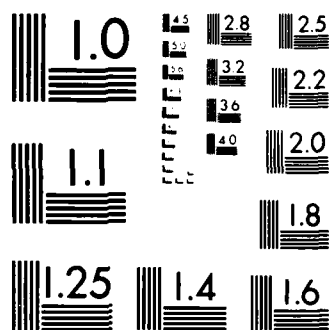
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PERIODIC SOLUTIONS OF PRESCRIBED ENERGY
FOR A CLASS OF HAMILTONIAN SYSTEMS

Vieri Benci and Paul H. Rabinowitz

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MATHEMATICS RESEARCH CENTER

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ABSTRACT

The main result of this paper is the following theorem: Let $p, q \in \mathbb{R}^n$, $H = H(p, q) \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ and let $H^{-1}(1)$ be the boundary of a compact neighborhood of 0 with $\nabla H \neq 0$ on $H^{-1}(1)$. If further $p \cdot H_p > 0$ on $H^{-1}(1)$ when $p \neq 0$, then the Hamiltonian system of ordinary differential equations

$$\dot{p} = -H_q(p, q), \quad \dot{q} = H_p(p, q)$$

possesses a periodic solution on $H^{-1}(1)$. The proof involves minimax arguments from the calculus of variations.

AMS (MOS) Subject Classifications: 34C15, 34C25

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Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

Hamiltonian systems of ordinary differential equations model the motion of a discrete mechanical system when no frictional forces are present. A basic property of such systems is that "energy" is conserved. Therefore solutions of Hamiltonian systems lie on surfaces of fixed energy. The main result of this paper is a fairly general criterion for such a surface to possess a periodic solution.

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PERIODIC SOLUTIONS OF PRESCRIBED ENERGY FOR A CLASS OF HAMILTONIAN SYSTEMS

Vieri Benci* and Paul H. Rabinowitz**

Introduction

Let $p, q \in \mathbb{R}^n$ and $H = H(p, q) : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be smooth. The problem to be studied here is the existence of periodic solutions of the associated Hamiltonian system of ordinary differential equations

$$(HS) \quad \dot{p} = -\frac{\partial H}{\partial q}(p, q), \quad \dot{q} = \frac{\partial H}{\partial p}(p, q)$$

where $\cdot \equiv \frac{d}{dt}$. Setting $z = (p, q)$, (HS) can also be written more succinctly as

$$\dot{z} = JH_z(z)$$

where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and I is the n dimensional identity matrix. As is well known any solution $z(t)$ of (HS) satisfies $H(z(t)) \equiv \text{constant}$, i.e. the "energy" H is an integral of the motion. Normalizing this constant to be 1, set $D \equiv H^{-1}(1)$. For $\xi, \eta \in \mathbb{R}^j$, $\xi \cdot \eta$ will denote their inner product. Our main result is

Theorem 1: Suppose H satisfies

$$(H_1) \quad H \in C^2(\mathbb{R}^{2n}, \mathbb{R}),$$

(H₂) D is the boundary of a compact neighborhood of 0 and $H_z \neq 0$ on D (i.e. D is a manifold).

$$(H_3) \quad p \cdot H_p \neq 0 \text{ if } p \neq 0.$$

Then (HS) possesses a periodic solution on D .

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Theorem 1 has several predecessors. Seifert [1] considered Hamiltonians of the form

$$H(p,q) = \sum_{i,j=1}^n a_{ij}(q)p_i p_j + V(q) ,$$

where B is a sum of kinetic and potential energy terms where $B \equiv \{q \in \mathbb{R}^n | V(q) < 1\}$ is diffeomorphic to the closed unit ball in \mathbb{R}^n and ∂B is a manifold, the matrix $(a_{ij}(q))$ is uniformly positive definite in B , and H is smooth. Using geodesic arguments from geometry, Seifert proved there exists a periodic solution of (HS) of a special type on ∂B . Generalizing his arguments, Weinstein [2] permitted a more general kinetic energy term $K(p,q)$ where for fixed q , K is even and convex in p while Gluck and Ziller [3] relaxed the condition on B merely requiring B to be compact with its boundary a manifold. See also Hayashi [4] and Benci [5] for results related to [3]. Another approach was made to (HS) in Rabinowitz [6] for $H = K + V$ where V satisfied Seifert's condition and $p \cdot K_p > 0$ for $p \neq 0$. A case not covered by Theorem 1 but which can be obtained by similar but simpler arguments was given in [7] in which ∂ is the boundary of a compact star-shaped neighborhood of 0.

In a different direction from Theorem 1, there has been some recent work on the multiplicity of solutions of (HS) on ∂ , generally when ∂ bounds a convex region in \mathbb{R}^{2n} . See e.g. Ekeland-Lasry [8], Ambrosetti-Mancini [9], van Groesen [10], Berestycki-Lasry-Mancini-Ruf [11], and Ekeland [12].

We will prove Theorem 1 by a direct variational approach using minimax arguments. The proof relies in part on ideas from [5-6]. Let $z(t) = (p(t), q(t))$ be 2π periodic and

$$A(z) \equiv \int_0^{2\pi} p \cdot \dot{q} dt .$$

A solution of (HS) will be obtained as a critical point of A restricted to $M \equiv \Psi^{-1}(1)$

$$\bar{V}(z) \equiv \frac{1}{2\pi} \int_0^{2\pi} \bar{H}(z) dt ,$$

$\bar{H}(z) = H(z)$ on D and is suitably modified on $\mathbb{R}^{2n} \setminus D$. This critical point is produced as a minimax of $A|_M$ over an appropriate class of subsets of M . In this approach, the unknown period appears as a Lagrange multiplier.

The modified Hamiltonian \bar{H} will be defined in §1 where some simple corollaries of Theorem 1 will also be obtained. In §2 the functional analytical framework in which the problem is treated is introduced. The properties of M and $A|_M$ such as the Palais-Smale condition are dealt with in §3. Theorem 1 is proved in §4. A dual variational argument is used in §5 to give an alternate approach to Theorem 1. In §6 a priori bounds from above and below are obtained for the unknown period of any solution of (HS) in terms of $A(z)$. Lastly in §7, the results of §4 and 6 are used to prove a stronger version of Theorem 1 with (H_1) replaced by

$$(H_1') \quad H \in C^1(\mathbb{R}^{2n}, \mathbb{R}) .$$

An intriguing open question concerning (HS) is whether Theorem 1 remains true or is false if hypothesis (H_3) is omitted.

§1. The Modified Hamiltonian

For technical reasons that will become clear later, the Hamiltonian will be redefined outside of a neighborhood of \mathcal{D} . Suppose H satisfies $(H_1)-(H_3)$. Then $H(0) < 1$ and $H > 1$ outside of the neighborhood of 0 bounded by \mathcal{D} . Without loss of generality we can assume $H(0) < \frac{1}{2}$. Our initial modification of H will allow us to assume $H > 0$, is a multiple of $|z|^2$ near 0 , satisfies $(H_1)-(H_3)$, and H_{zz} is uniformly bounded.

Indeed since $H(0) < \frac{1}{2}$, $\rho > 0$ can be chosen so that $\rho|z|^2 > \frac{3}{2}$ if $H(z) > 1$ and $\rho^{-1}|z|^2 < \frac{1}{4}$ if $H(z) < 1$. Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(s) = 0$ if $s \leq \frac{1}{4}$; $\chi(s) = 1$ if $s > \frac{1}{2}$, and $\chi'(s) > 0$ if $s \in (\frac{1}{4}, \frac{1}{2})$. Define

$$\tilde{H}(z) = \chi(H(z) - 1)[\rho|z|^2 - H(z)] + H(z) + \chi(1 - H(z))[\rho^{-1}|z|^2 - H(z)].$$

Then $\tilde{H} \in C^2(\mathbb{R}^{2n}, \mathbb{R})$, $\tilde{H} = \rho^{-1}|z|^2$ near $z = 0$, and $\tilde{H} = H$ near \mathcal{D} . Moreover if $\tilde{H}(z) = 1$, then $z \in \mathcal{D}$. To see this, suppose $H(z) > 1$. Then

$$\tilde{H}(z) = \chi(H(z) - 1)[\rho|z|^2 - H(z)] + H(z).$$

If $H(z) > \frac{3}{2}$, $\tilde{H}(z) = \rho|z|^2 > \frac{3}{2} > 1$ while if $H(z) \in (1, \frac{3}{2})$, $\tilde{H}(z) > H(z) > 1$. Similar reasoning shows $\tilde{H}(z) < 1$ if $H(z) < 1$. Thus $\tilde{H}^{-1}(1) = \mathcal{D}$. A related argument shows

$\tilde{H}(z) > 0$ if $z \neq 0$. It is clear that \tilde{H} satisfies $(H_1)-(H_2)$. To verify (H_3) , by the definition of χ , it suffices to show that for $p \neq 0$,

$$\begin{aligned} (1.1) \quad p \cdot \tilde{H}_p(z) &= \{\chi'(H(z) - 1)[\rho|z|^2 - H(z)] - \chi'(1 - H(z))[\rho^{-1}|z|^2 - H(z)]\}p \cdot H_p(z) \\ &\quad + (1 - \chi(H(z) - 1) - \chi(1 - H(z)))p \cdot H_p(z) \\ &\quad + 2|p|^2[\chi(H(z) - 1)\rho + \chi(1 - H(z))\rho^{-1}] > 0 \end{aligned}$$

if $z \in H^{-1}[\frac{1}{2}, \frac{3}{2}]$. Again this follows from our choice of ρ and (H_3) .

Since \mathcal{D} is compact, there is a $\beta > 0$ such that $|z| \leq \beta$ for $z \in \mathcal{D}$. Let $\hat{\chi} \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\hat{\chi}(s) = 1$ for $s \leq 2\beta$, $\hat{\chi}(s) = 0$ for $s \geq 4\beta$, and $\hat{\chi}'(s) < 0$ for $s \in (2\beta, 4\beta)$. Set

$$\hat{H}(z) = \hat{\chi}(|z|)\tilde{H}(z) + (1 - \hat{\chi}(|z|))\hat{\rho}|z|^2.$$

Then it is easy to check that for $\hat{\rho}$ chosen so that $\hat{\rho}|z|^2 > \tilde{H}(z)$ for $|z| \in (2\beta, 4\beta)$, \hat{H}

possesses the properties verified above for \tilde{H} , \hat{H} is a multiple of $|z|^2$ for large $|z|$, and satisfies

(H₄) $H_{zz}(z)$ is uniformly bounded.

Remark 1.2: The above arguments work equally well if H merely satisfies (H₁) and (H₃) in a neighborhood of \mathcal{D} .

Next hypothesis (H₃) will be used to decompose H into a sum of kinetic and potential energy terms. Set $U(q) = \hat{H}(0, q)$ and $K(p, q) \equiv K(z) \equiv \hat{H}(z) - U(q)$. Note that $K(z) > 0$ via (H₃) and $K, U \in C^2$ via (H₁). Moreover

Proposition 1.3: K satisfies the following properties:

(K₁) $K(0, q) = 0$

(K₂) $p \cdot K_p(z) > 0$ if $p \neq 0$

(K₃) $|K_p(z)| \leq a_1(1 + |z|)$

(K₄) $K(z) \leq a_1(1 + |z|)|p|$

(K₅) $|K_q(z)| \leq a_2|p|$

(K₆) $K_{zz}(z)$ is uniformly bounded in \mathbb{R}^{2n} .

(In (K₃)-(K₅) and later, a_1 denotes a constant.)

Proof: (K₁) and (K₂) follow from the definition of K and (H₃), and (K₃), (K₆) from (H₄). Since

$$(1.4) \quad K(p, q) = \int_0^1 \frac{d}{ds} K(sp, q) ds = \int_0^1 p \cdot K_p(sp, q) ds,$$

(K₃) and (1.4) imply (K₄). Similarly

$$(1.5) \quad K_q(p, q) = H_q(p, q) - H_q(0, q) = \int_0^1 H_{pq}(sp, q) p ds$$

so (H₄) and (1.5) give (K₅).

To define \bar{H} , one final modification of H is required. Let

$$\Omega_s = \{q \in \mathbb{R}^n | U(q) < s\}.$$

By (H_2) , $U_q \neq 0$ on $\partial\Omega_1$ and there exist constants $d, \beta > 0$ such that $U_q(q) \neq 0$ and

$$(1.6) \quad |U_q(q)| > \beta U(q) \quad \text{if } q \in \Omega_{1+2d} \setminus \Omega_{1-2d}.$$

Let $\phi \in C^2$ be defined for $s < 1 + 2d$ such that

$$(\phi_1) \quad \phi(s) = s, \quad s < 1 + d$$

$$(\phi_2) \quad \phi'(s) > 1, \quad s < 1 + 2d$$

$$(\phi_3) \quad \phi(s) = (s - (1 + 2d))^{-2}, \quad s \text{ near } 1 + 2d.$$

We further extend ϕ to all of \mathbb{R} via $\phi(s) = \infty$ if $s > 1 + 2d$. Finally define

$$V(q) \equiv \phi(U(q)) \quad \text{for } q \in \mathbb{R}^n \text{ and}$$

$$\bar{H}(z) \equiv K(z) + V(q).$$

Thus \bar{H} is C^2 where finite and if $\bar{H}(z) = 1$, $U(q) < V(q) < 1$ by (ϕ_2) . Thus

$$V(q) = U(q) \quad \text{by } (\phi_1) \quad \text{and} \quad \bar{H}(z) = H(z). \quad \text{Consequently } \bar{H}^{-1}(1) = \mathcal{D}.$$

We will find a periodic solution of

$$(1.7) \quad \dot{p} = -\bar{H}_q, \quad \dot{q} = \bar{H}_p$$

on $\bar{H}^{-1}(1)$. Hence it will be a periodic solution of (HS) on \mathcal{D} .

To conclude this section, some estimates will be obtained for V . Let $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\begin{aligned} \psi(s) &= 0, & s < 1 - 2d \\ &= 1, & s > 1 - d \end{aligned}$$

and $\psi'(s) > 0$ if $s \in (1 - 2d, 1 - d)$. Set

$$v(q) = \psi(V(q)) \frac{U_q(q)}{|U_q(q)|}.$$

Observing that the ψ term vanishes if $q \in \Omega_{-2d}$ and $\psi'(V(q))$ vanishes if $V(q) > 1 - d$, it follows that $v \in C^1(\mathbb{R}^n, \mathbb{R})$.

Proposition 1.8: There is a constant $\gamma > 0$ such that

$$(1.9) \quad v_q(q) \cdot v(q) > 0 \quad \text{for all } q \in \Omega_{1+2d}$$

and

$$(1.10) \quad v_q(q) \cdot v(q) > \gamma V(q) \quad \text{for all } q \in \Omega_{1+2d} \setminus \Omega_{1-2d}.$$

Proof: Inequality (1.9) is immediate from the definition of V and (ϕ_2) . To check (1.10), note that if $U(q) \in (1-d, 1+2d)$, then $V(q) \geq U(q)$ by (ϕ_2) and $\psi(V(q)) = 1$. Therefore

$$(1.11) \quad v_q(q) \cdot v(q) = \phi'(U(q)) |U_q| \geq \beta(1-d)\phi'(U(q))$$

by (1.6). Thus to get (1.10), it suffices to show

$$(1.12) \quad \phi'(s) \geq \beta_1 \phi(s), \quad s \in (1-d, 1+2d)$$

and this is immediate from (ϕ_1) - (ϕ_3) .

§2. Functional Analytical Preliminaries

The space in which (1.7) will be treated is the Hilbert space

$$E \equiv \{z = (p, q) \mid p \in L^2(S^1, \mathbb{R}^n), q \in W^{1,2}(S^1, \mathbb{R}^n)\} \\ \equiv L^2(S^1, \mathbb{R}^n) \oplus W^{1,2}(S^1, \mathbb{R}^n)$$

where $L^2(S^1, \mathbb{R}^n)$ denotes the set of n -tuples of 2π periodic functions which are square integrable, etc. For $w \in L^2(S^1, \mathbb{R}^n)$, let

$$[w] \equiv \frac{1}{2\pi} \int_0^{2\pi} w(t) dt.$$

Thus any $z = (p, q) \in E$ can be decomposed into $([p], [q]) + (\hat{p}, \hat{q})$ where $(\hat{p}, \hat{q}) \in \hat{L}^2 \oplus \hat{W}^{1,2}$ and

$$\hat{L}^2 \equiv \{p \in L^2(S^1, \mathbb{R}^n) \mid [p] = 0\}, \\ \hat{W}^{1,2} \equiv \{q \in W^{1,2}(S^1, \mathbb{R}^n) \mid [q] = 0\}.$$

As inner product in E we take

$$(z_1, z_2)_E \equiv \int_0^{2\pi} [(\hat{p}_1(t) \cdot \hat{p}_2(t)) + (D\hat{q}_1 \cdot D\hat{q}_2)] dt + [p_1][p_2] + [q_1][q_2]$$

where $D \equiv \frac{d}{dt}$ and $z_1 = (p_1, q_1) = ([p_1] + \hat{p}_1, [q_1] + \hat{q}_1)$, etc. The norm in E will be denoted by $\|\cdot\|$ and we will generally use the same notation for the norm in E^* , the dual space of E .

It is easy to see that $D|_{\hat{W}^{1,2}} : \hat{W}^{1,2} \rightarrow \hat{L}^2$ is an isomorphism. Let D^{-1} denote its inverse. We define linear maps P^0, P^+, P^- of E into E by

$$P^0(p, q) \equiv ([p], [q])$$

and

$$P^\pm(p, q) \equiv \left(\frac{1}{2}(\hat{p} \pm D\hat{q}), \frac{1}{2}(\hat{q} \pm D^{-1}\hat{p})\right).$$

It is easy to verify that these maps are well defined and are (continuous) projectors on

E satisfying $p^0 + p^+ + p^- = \text{id}$, the identity map on E . Define $E^0 \equiv p^0 E$ and

$E^\pm \equiv p^\pm E$. Note that if $(p^\pm, q^\pm) \in E^\pm$, then $p^\pm = \frac{1}{2} (p^\pm \pm Dq^\pm)$. Therefore

$$(2.1) \quad p^\pm = \pm Dq^\pm.$$

Next observe that the spaces E^0, E^\pm are mutually orthogonal subspaces of E . E.g. if $z^\pm = (p^\pm, q^\pm) \in E^\pm$,

$$\begin{aligned} (z^+, z^-)_E &= \int_0^{2\pi} [(p^+ \cdot p^-) + (Dq^+ \cdot Dq^-)] dt \\ &= \int_0^{2\pi} [(Dq^+ \cdot (-Dq^-)) + (Dq^+ \cdot Dq^-)] dt = 0 \end{aligned}$$

via (2.1).

For $z = (p, q) \in E$, define the action integral as

$$A(z) \equiv \int_0^{2\pi} p \cdot \dot{q} \, dt.$$

Then $A \in C^\infty(E, \mathbb{R})$ and writing $z = z^0 + z^+ + z^-$ and using (2.1) shows

$$\begin{aligned} (2.2) \quad A(z) &= \int_0^{2\pi} (p^0 + p^+ + p^-) \cdot (Dq^+ + Dq^-) dt \\ &= \int_0^{2\pi} [(p^+ \cdot Dq^+) + (p^+ \cdot Dq^-) + (p^- \cdot Dq^+) + (p^- \cdot Dq^-)] dt \\ &= \frac{1}{2} \int_0^{2\pi} (|p^+|^2 + |Dq^+|^2 - |p^-|^2 - |Dq^-|^2) dt \\ &= \frac{1}{2} (\|z^+\|^2 - \|z^-\|^2). \end{aligned}$$

Next define

$$\Psi(z) \equiv \frac{1}{2\pi} \int_0^{2\pi} H(z) dt$$

and

$$M \equiv \{z \in E \mid \Psi(z) = 1\}.$$

Our goal is to obtain a periodic solution of (HS) (or equivalently (1.7)) as a critical point of $A|_M$. As will be seen later, a critical point z of this constrained variational problem satisfies $z \in C^1(S^1, \mathbb{R}^{2n})$ and

$$(2.3) \quad \dot{z} = \lambda J \bar{H}_z(z)$$

where $\lambda \in \mathbb{R} \setminus \{0\}$. Since (2.3) is a Hamiltonian system, $\bar{H}(z(t)) \equiv \text{constant}$. Thus $z \in M$ implies $z \in \mathcal{D}$. Moreover since $\lambda \neq 0$, rescaling time in (2.3) yields a periodic solution of (1.7) on \mathcal{D} , i.e. the desired solution of (HS).

§3. Some Properties of M and $A|_M$

This section studies several properties of Ψ and M . It will be shown that M is a $C^{1,1}$ manifold which bounds a neighborhood of 0 in E and $A|_M$ satisfies a version of the Palais-Smale condition.

For $q \in W^{1,2}(S^1, \mathbb{R}^n)$, set

$$V(q) \equiv \frac{1}{2\pi} \int_0^{2\pi} v(q(t)) dt.$$

Remark 3.1: The definition of V shows that if $V(q) < \infty$, $q(t) \in \Omega_{1+2d}$ for almost all $t \in [0, 2\pi]$ and since $q \in C(S^1, \mathbb{R}^n)$, $q(t) \in \bar{\Omega}_{1+2d}$ for all $t \in [0, 2\pi]$. In particular there is a constant $M > 0$ (and independent of q) such that $\|q\|_L \leq M$.

For $x \in \mathbb{R}^n$, let

$$l(x) \equiv \inf_{y \in \partial\Omega_{1+2d}} |x - y|.$$

Proposition 3.2: There exist constants β_1, M_1 such that for all $x \in \bar{\Omega}_{1+2d}$,

$$(3.3) \quad \beta_1 l(x)^{-2} \leq v(x) + M_1.$$

Proof: Let $x \in \bar{\Omega}_{1+2d}$. Using the implicit function theorem, it is not difficult to show that there is an $\epsilon_0 > 0$ such that if $l(x) < \epsilon_0$, there exists a unique $\bar{x} \in \partial\Omega_{1+2d}$ and $\rho > 0$ such that

$$x = \bar{x} - \rho U_q(\bar{x}).$$

Therefore there is a $\beta_2 > 0$ such that

$$(3.4) \quad \beta_2 \rho > |x - \bar{x}| > \beta \rho \quad \text{if } l(x) < \epsilon_0$$

via the continuity of U_q and (1.6). Now

$$(3.5) \quad U(x) = U(\bar{x}) = U_q(\bar{x})(x - \bar{x}) + o(|x - \bar{x}|)$$

as $x \rightarrow \partial\Omega_{1+2d}$. Therefore by (3.4) - (3.5), for x near $\partial\Omega_{1+2d}$, e.g. $l(x) < \epsilon$,

$$(3.6) \quad |U(x) - U(\bar{x})| \leq \rho |U_q(\bar{x})|^2 + o(\rho) \leq M_2 \rho.$$

Now for $l(x) < \epsilon$, by (ϕ_3) and (3.6),

$$v(x) = (U(x) - (1 + 2d))^{-2} > (M_2 \rho)^{-2}.$$

But $\rho = \ell(x) |U_q(\bar{x})|^{-1}$. Therefore

$$(3.7) \quad V(x) > M_3 \ell(x)^{-2}$$

if $\ell(x) < \varepsilon$. If $\ell(x) > \varepsilon$, $\ell(x)^{-2} < \varepsilon^{-2}$ so

$$V(x) + M_3 \varepsilon^{-2} > M_3 \ell(x)^{-2}.$$

Thus (3.3) obtains with $M_2 = M_3 \varepsilon^{-2}$ and $\beta_1 = M_3$.

The estimate (3.3) will be used next to show that $V(q) < \infty$ implies that $q(t)$ avoids $\partial\Omega_{1+2d}$.

Proposition 3.8: Let $q \in W^{1,2}(S^1, \mathbb{R}^n)$ satisfy $V(q) < \infty$. Then there is an

$$\bar{M} = \bar{M}(\|q\|_{W^{1,2}}, V(q)) > 0 \text{ such that } \ell(q(t)) > \bar{M} \text{ for all } t \in [0, 2\pi].$$

Proof: Since q is 2π periodic, by translating t it can be assumed that

$$\ell(q(0)) = \min_{t \in [0, 2\pi]} \ell(q(t)) \equiv \mu.$$

By the Cauchy-Schwarz inequality,

$$(3.9) \quad |q(t) - q(0)| \leq \int_0^t |\dot{q}(\tau)| d\tau \leq t^{1/2} \|q\|_{W^{1,2}}.$$

Since ℓ is Lipschitz continuous (with constant 1),

$$(3.10) \quad |\ell(q(t)) - \ell(q(0))| \leq |q(t) - q(0)| \leq t^{1/2} \|q\|_{W^{1,2}}.$$

Therefore

$$(3.11) \quad \ell(q(t)) \leq \mu + t^{1/2} \|q\|_{W^{1,2}}.$$

We can assume $\|q\|_{W^{1,2}} > 0$ for otherwise the result is trivial. By (3.3) and (3.11),

$$\begin{aligned}
(3.12) \quad \infty > (q) &> \frac{1}{2\pi} \int_0^{2\pi} (\beta_1 \ell(q(t))^{-2} - M_1) dt \\
&> \frac{\beta_1}{2\pi} \int_0^{2\pi} (\mu + t^{1/2} \|q\|_{W^{1,2}})^{-2} dt - M_1 \\
&> \frac{\beta_1}{\pi} \int_0^{2\pi} (\mu^2 + t \|q\|_{W^{1,2}}^2)^{-1} dt - M_1 \\
&= \frac{\beta_1}{\pi \|q\|_{W^{1,2}}^2} \log\left(1 + \frac{2\pi \|q\|_{W^{1,2}}^2}{\mu^2}\right) - M_1
\end{aligned}$$

from which the result follows.

Remark 3.13: Proposition 3.8 implies that the domain of V is

$$\{q \in W^{1,2}(S^1, \mathbb{R}^n) \mid q(t) \in \Omega_{1+2d} \text{ for all } t \in [0, 2\pi]\}.$$

The smoothness of V will be established next.

Proposition 3.14: $V \in C^2$ on the domain of V .

Proof: Let $q \in W^{1,2}(S^1, \mathbb{R}^n)$ with $V(q) < \infty$. Let $\delta = \inf_{t \in [0, 2\pi]} \ell(q(t))$. Then $\delta > 0$ by Proposition 3.8. Let $\tilde{q} \in W^{1,2}(S^1, \mathbb{R}^n)$ with $\|\tilde{q}\|_{W^{1,2}} < \rho$. If ρ is sufficiently small, $\|\tilde{q}\|_{L^\infty} < \delta/2$. Therefore since $v \in C^2(\Omega_{1+2d}, \mathbb{R})$,

$$(3.15) \quad V(q + \tilde{q}) = V(q) + v_q(q)\tilde{q} + \frac{1}{2} v_{qq}(q)(\tilde{q}, \tilde{q}) + o(|\tilde{q}|^2)$$

as $\tilde{q} \rightarrow 0$ uniformly for $t \in [0, 2\pi]$. The definition of V , (3.15), and the compact embedding of $W^{1,2}(S^1, \mathbb{R}^n)$ in $C(S^1, \mathbb{R}^n)$ then readily imply that V is Frechet differentiable at q with

$$V'(q)\tilde{q} = \frac{1}{2\pi} \int_0^{2\pi} v'(q)\tilde{q} dt,$$

$V'(q)$ is continuous, $(V'(q))' \equiv V''(q)$ exists,

$$V''(q)(\tilde{q}, \tilde{q}) = \frac{1}{2\pi} \int_0^{2\pi} v''(q)(\tilde{q}, \tilde{q}) dt$$

and $V''(q)$ is continuous.

For $z \in E$, set

$$K(z) \equiv \frac{1}{2\pi} \int_0^{2\pi} K(z) dt .$$

Proposition 3.16: $K \in C^{1,1}(E, \mathbb{R})$, (i.e. K is Frechet differentiable and its Frechet derivative is Lipschitz continuous).

Proof: Since $K \in C^2(\mathbb{R}^{2n}, \mathbb{R})$, given any $z, \zeta \in \mathbb{R}^{2n}$, by Taylor's Theorem,

$$(3.17) \quad K(z + \zeta) = K(z) + K_z(z)\zeta + \frac{1}{2} K_{zz}(z + \theta\zeta)(\zeta, \zeta)$$

for some $\theta \in (0, 1)$. By (K_6) of Proposition 1.3, K_{zz} is uniformly bounded. Therefore there is a constant $M_3 > 0$ such that

$$(3.18) \quad |K(z + \zeta) - K(z) - K_z(z)\zeta| \leq M_3 |\zeta|^2$$

for all $z, \zeta \in \mathbb{R}^{2n}$. Choosing $z, \zeta \in E$, (3.18) implies

$$(3.19) \quad |K(z + \zeta) - K(z) - \frac{1}{2\pi} \int_0^{2\pi} K_z(z)\zeta dt| \leq \frac{M_3}{2\pi} \|\zeta\|_2^2 \leq M_4 \|\zeta\|^2 .$$

In particular for z fixed, given any $\varepsilon > 0$, if ζ is sufficiently small, the right hand side of (3.19) does not exceed $\varepsilon \|\zeta\|$. Hence K is Frechet differentiable and

$$K'(z)\zeta = \frac{1}{2\pi} \int_0^{2\pi} K_z(z)\zeta dt .$$

To show that K' is Lipschitz continuous, note that

$$(3.20) \quad \|K'(z+w) - K'(z)\|_* = \sup_{\zeta \in E, \|\zeta\| \leq 1} \left| \frac{1}{2\pi} \int_0^{2\pi} (K_z(z+w) - K_z(z))\zeta dt \right| .$$

As in (3.17) by the Mean Value Theorem and (K_6) ,

$$(3.21) \quad |K_z(z+w) - K_z(z)| \leq M_5 |w|$$

for some constant M_5 . Therefore (3.20)-(3.21) imply

$$(3.22) \quad \|K'(z+w) - K'(z)\|_E \leq M_6 \|w\|, \quad E$$

i.e. K' is Lipschitz continuous.

By the definitions of K and V , $\Psi = K + V$. Recall that $M = \Psi^{-1}(1)$. The next three propositions study some properties of M .

Proposition 3.23: M is $C^{1,1}$ manifold in E .

Proof: The smoothness assertions follow on combining Proposition 3.14 and 3.16 once we show that M is a manifold, i.e. $\Psi'(z) \neq 0$ for all $z \in M$. But if $z = (p, q)$ with $p \neq 0$,

$$\Psi'(z)(p, 0) = \frac{1}{2\pi} \int_0^{2\pi} p \cdot K_p(z) dt > 0$$

via (K_2) of Proposition 1.3. If $p \equiv 0$, then $\Psi(z) = V(q) = 1$ via (K_1) and by Proposition 3.8, $q(t) \in \Omega_{1+2d}$ for all $t \in [0, 2\pi]$. If $q(t) \in \Omega_{1-d}$ for all $t \in [0, 2\pi]$, $V(q) < 1 - d$ which is impossible. Thus since $q \in C(S^1, \mathbb{R}^n)$, $q(t) \in \Omega_{1+2d} \setminus \Omega_{1-d}$ on a set Y of positive measure. By previous remarks, $v(q) \in C(S^1, \mathbb{R}^n)$ and $(0, v(q)) \in E$. Hence by (1.9)-(1.10),

$$(3.24) \quad \Psi'(z)(0, v(q)) = \int_Y v(q) \cdot \gamma(1-d) dt > 0.$$

Thus M is a manifold and the proof is complete.

Proposition 3.25: M is the boundary of a neighborhood of 0 in E .

Proof: Ψ is continuous on $L^2(S^1, \mathbb{R}^n) \oplus \{q \in W^{1,2}(S^1, \mathbb{R}^n) | V(q) < \infty\}$ which is an open set in E . Therefore $\Psi^{-1}(-\infty, 1)$ is open. Since $\bar{H}(0) = 0$, 0 belongs to this set.

Proposition 3.26: M is bounded in $L^2(S^1, \mathbb{R}^{2n})$.

Proof: Let $(p, q) \in M$. Since $V(q) < \infty$, by Remark 3.1, there is an $M > 0$ such that $\|q\|_L^\infty < M$. The definition of K implies it is a multiple of $|p|^2$ for $|z|$ near ∞ . Hence there are constants $M_7, M_8 > 0$ such that

$$K(z) > M_7 |p|^2 - M_8$$

for all $z \in \mathbb{R}^{2n}$. Therefore

$$K(z) > \frac{M_7}{2\pi} \|p\|_{L^2}^2 - \frac{M_8}{2\pi}.$$

Thus if $z \in M$,

$$\|p\|_{L^2}^2 < \left(1 + \frac{M_8}{2\pi}\right) \frac{2\pi}{M_7}.$$

The next proposition shows that $A|_M$ satisfies a version of the Palais-Smale condition.

Proposition 3.27: $A|_M$ satisfies $(PS)^+$, i.e. if $c > 0$ and (z_j) is a sequence in E such that

(i) $z_j \in M$,

(ii) $A(z_j) \rightarrow c$,

and (iii) $A'(z_j) - \lambda_j \Psi'(z_j) \rightarrow 0$ (in E^*)

as $j \rightarrow \infty$ where

$$\lambda_j = (A'(z_j), \Psi'(z_j))_{E^*, E} \|\Psi'(z_j)\|^{-2}$$

then (z_j) has a convergent subsequence.

Proof: By (ii), there is an $\varepsilon \in (0, \frac{c}{2})$ such that

$$(3.28) \quad c - \varepsilon \leq A(z_j) \leq c + \varepsilon$$

for all large $j \in \mathbb{N}$. Similarly by (iii), there is a $w_j \in E^*$ with $w_j \rightarrow 0$ as $j \rightarrow \infty$

and

$$(3.29) \quad A'(z_j)\zeta - \lambda_j \Psi'(z_j)\zeta = \langle w_j, \zeta \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between E^* and E . Choosing $\zeta = (p_j, 0)$ yields

$$(3.30) \quad A(z_j) - \frac{\lambda_j}{2\pi} \int_0^{2\pi} K_p(z_j) p_j dt = \langle w_j, (p_j, 0) \rangle.$$

By (3.28) and the choice of ε , $A(z_j) > \frac{c}{2}$ while the left hand side of (3.30) goes to 0

as $j \rightarrow \infty$ since $((p_j, 0))$ is bounded in E . Consequently by $(K_2)-(K_3)$ of Proposition 1.3, λ_j is positive and bounded away from 0 for large j . We restrict ourselves to such $j \in \mathbb{N}$.

Choosing $\zeta = (\xi, \eta)$, (3.29) can be rewritten as

$$(3.31) \quad \int_0^{2\pi} [(p_j \cdot \dot{h}) + (\xi \cdot \dot{q}_j)] dt - \frac{\lambda_j}{2\pi} \int_0^{2\pi} [(\bar{H}_p(z_j) \cdot \xi) + (\bar{H}_q(z_j) \cdot \eta)] dt = \langle w_j, \zeta \rangle.$$

Setting $w_j = (u_j, v_j) \in E^*$, i.e. $u_j \in L^2(S^1, \mathbb{R}^n)$ and $v_j \in W^{-1,2}(S^1, \mathbb{R}^n)$ and noting that $\dot{p}_j \in W^{-1,2}(S^1, \mathbb{R}^n)$, (3.31) implies that

$$(3.32) \quad (i) \quad -\dot{p}_j = \frac{\lambda_j}{2\pi} \bar{H}_q(z_j) + v_j$$

$$(ii) \quad \dot{q}_j = \frac{\lambda_j}{2\pi} \bar{H}_p(z_j) + u_j = \frac{\lambda_j}{2\pi} K_p(z_j) + u_j$$

holds in the sense of distributions.

We claim that

$$(3.33) \quad \int_0^{2\pi} K_p(z_j) \cdot p_j dt \geq a > 0$$

for all $j \in \mathbb{N}$. Assuming (3.33) for now, (3.30) and (3.28) then imply λ_j is bounded away from ∞ . Then by (3.32) (ii),

$$(3.34) \quad \|\dot{q}_j\|_{L^2} \leq \frac{\lambda_j}{2\pi} \|K_p(z_j)\|_{L^2} + \|u_j\|_{L^2}$$

so (K_3) of Proposition 1.3, the boundedness of (z_j) in L^2 (via Proposition 3.26) and Remark 3.1 show that (q_j) is bounded in $W^{1,2}(S^1, \mathbb{R}^n)$. Proposition 3.8 then implies there is an $\bar{M} > 0$ independent of j such that $l(q_j(t)) \geq \bar{M}$ for all $t \in [0, 2\pi]$, i.e. the functions q_j lie uniformly inside Ω_{1+2d} . Therefore the functions $(V_q(q_j))$ are bounded in L^∞ and by (K_5) of Proposition 1.3, $\bar{H}_q(z_j)$ are bounded in L^2 . Consequently

the right hand side of (3.32)(i) converges strongly in $W^{-1,2}(S^1, \mathbb{R}^n)$ (along a subsequence). Therefore \dot{p}_j converges strongly in $W^{-1,2}(S^1, \mathbb{R}^n)$. Consequently $D^{-1}\dot{p}_j$ converges strongly in $L^2(S^1, \mathbb{R}^n)$. But $p_j = [p_j] + D^{-1}\dot{p}_j$. Hence along a subsequence p_j converges strongly in $L^2(S^1, \mathbb{R}^n)$. Lastly by (3.32) (ii), the same is true for a subsequence of q_j in $W^{1,2}(S^1, \mathbb{R}^n)$ since (K_3) of Proposition 1.3 implies that K_p is a continuous map of $L^2(S^1, \mathbb{R}^{2n})$ to $L^2(S^1, \mathbb{R}^n)$.

Thus Proposition 3.27 will be established once we show that (3.33) holds. Suppose that (3.33) is false, i.e.

$$(3.35) \quad \int_0^{2\pi} K_p(z_j) \cdot p_j dt \rightarrow 0$$

for some subsequence of j 's. This implies that $\|p_j\|_{L^2} \rightarrow 0$ along this subsequence. Indeed let $Y_{1j} = \{t \in [0, 2\pi] \mid |p_j(t)| < \sigma\}$, $Y_{2j} = \{t \in [0, 2\pi] \mid \sigma < |p_j(t)| < 4\beta\}$, and $Y_{3j} = \{t \in [0, 2\pi] \mid |p_j(t)| > 4\beta\}$ where $\sigma < 1$ and β was defined together with \hat{H} in §1. On Y_{3j} , $p_j \cdot K_p(z_j) = 2\tilde{\rho}|p_j|^2$. On Y_{2j} , both $|p|^2$ and $p \cdot K_p$ are bounded away from 0. Therefore there is a constant $\alpha = \alpha(\sigma)$ such that $|p|^2 < \alpha(\sigma)p \cdot K_p$ on Y_{2j} . Combining these observations yields

$$(3.36) \quad \int_0^{2\pi} |p_j|^2 dt < 2\pi\sigma^2 + \alpha(\sigma) \int_{Y_{2j}} p_j \cdot K_p(z_j) dt \\ + (2\hat{\rho})^{-1} \int_{Y_{3j}} p_j \cdot K_p(z_j) dt.$$

Since σ is arbitrary, (3.35)-(3.36) show $\|p_j\|_{L^2} \rightarrow 0$ as $j \rightarrow \infty$. Then (K_3) -(K_4) of Proposition 1.3 imply

$$(3.37) \quad K(z_j) \rightarrow 0$$

as $j \rightarrow \infty$ while (K_3) and Proposition 3.26 show

$$(3.38) \quad \int_0^{2\pi} |K_p(z_j)| |p_j| dt \rightarrow 0$$

as $j \rightarrow \infty$. Also by (K_5) ,

$$(3.39) \quad \left| \int_0^{2\pi} K_q(z_j) \cdot v(q_j) dt \right| \leq \|K_q(z_j)\|_{L^2} \|v(q_j)\|_{L^2} \rightarrow 0$$

as $j \rightarrow \infty$ since $(\|v(q_j)\|_{L^2})$ is uniformly bounded. By (3.37) and (i) of Proposition 3.27,

$$(3.40) \quad V(q_j) \rightarrow 1$$

as $j \rightarrow \infty$. Set

$$\Delta_j \equiv \{t \in [0, 2\pi] \mid V(q_j(t)) > 1 - d\}.$$

Therefore

$$(3.41) \quad \begin{aligned} V(q_j) &\leq \frac{1}{2\pi} \left(\int_{\Delta_j} V(q_j(t)) dt + \int_{[0, 2\pi] \setminus \Delta_j} V(q_j(t)) dt \right) \\ &\leq \frac{1}{2\pi} \int_{\Delta_j} V(q_j(t)) dt + 1 - d. \end{aligned}$$

Hence for large j , by (3.40)-(3.41),

$$(3.42) \quad \frac{d}{2} \leq \frac{1}{2\pi} \int_{\Delta_j} V(q_j(t)) dt.$$

Next by (3.32) (i) and (1.9),

$$(3.43) \quad \begin{aligned} \int_0^{2\pi} p_j \cdot \frac{d}{dt} v(q_j(t)) dt &= \frac{\lambda_j}{2\pi} \int_0^{2\pi} \bar{H}_q(z_j) \cdot v(q_j) dt + \langle v_j, v(q_j) \rangle \\ &> \frac{\lambda_j}{2\pi} \left[\int_{\Delta_j} v_q(q_j) \cdot v(q_j) dt - \|K_q(z_j)\|_{L^2} \|v(q_j)\|_{L^2} \right] - \|v_j\|_{W^{-1,2}} \|v(q_j)\|_{W^{1,2}}. \end{aligned}$$

As has been noted earlier, $v(\cdot) \in C^1(\mathbb{R}^n, \mathbb{R})$ and $\|q\|_{L^\infty} < M$ for all $(p, q) \in M$. Hence there exists a constant $M_1 > 0$ and independent of j such that

$$\|v(q_j)\|_{W^{1,2}} \leq M_1 (1 + \|q_j\|_{L^2}).$$

Thus (3.42)-(3.43), (1.10), and (K_5) of Proposition 1.3 imply that

$$(3.44) \quad \int_0^{2\pi} p_j \cdot \frac{d}{dt} v(q_j) dt \geq \lambda_j \gamma \frac{d}{2} - o(1) - o(1) \|\dot{q}_j\|_{L^2}$$

as $j \rightarrow \infty$. On the other hand, by (3.32) and (3.38),

$$(3.45) \quad \begin{aligned} \int_0^{2\pi} p_j \cdot \frac{d}{dt} v(q_j) dt &\leq \|v'(q_j)\|_{L^\infty} \int_0^{2\pi} |p_j| |\dot{q}_j| dt \\ &\leq M_2(\lambda_j) \int_0^{2\pi} |K_p(z_j)| |p_j| dt + \int_0^{2\pi} |u_j| |p_j| dt \\ &\leq \lambda_j o(1) + o(1)^2 \end{aligned}$$

as $j \rightarrow \infty$. Combining (3.44)-(3.45) shows

$$(3.46) \quad \frac{\gamma d}{4} \lambda_j \leq o(1) \|\dot{q}_j\|_{L^2} + o(1)$$

as $j \rightarrow \infty$. But then by (3.32) (ii),

$$(3.47) \quad \frac{\gamma d}{4} \lambda_j \leq o(1) \lambda_j \|K_p(z_j)\|_{L^2} + o(1)$$

so (K_j) of Proposition 1.3 and Proposition 3.26 imply $\lambda_j \rightarrow 0$ as $j \rightarrow \infty$, a contradiction. Thus (3.33) has been verified and Proposition 3.27 has been established.

Two further technical results are needed in this section. Let L denote the duality map between $W^{-1,2}(S^1, \mathbb{R}^n)$ and $W^{1,2}(S^1, \mathbb{R}^n)$, i.e. L is defined by

$$(Lw, \xi)_{W^{1,2}} = \langle w, \xi \rangle$$

for $w \in W^{-1,2}(S^1, \mathbb{R}^n)$ and $\xi \in W^{1,2}(S^1, \mathbb{R}^n)$. Abusing notation somewhat, we will also let L denote the duality between E and E^* . For $z = (p, q) \in E$, define $P_1 z \equiv p$ and $P_2 z \equiv q$.

Proposition 3.48: $P_2 L \Psi'$ is a compact map of M into $W^{1,2}(S^1, \mathbb{R}^n)$.

Proof: For $z \in M$ and $\zeta = (u, v) \in E$,

$$\Psi'(z)\zeta = \langle \Psi'(z), \zeta \rangle = \frac{1}{2\pi} \int_0^{2\pi} [(\bar{H}_p(z) \cdot u) + (\bar{H}_q(z) \cdot v)] dt = (\zeta \Psi'(z), \zeta)_E.$$

Therefore

$$(3.49) \quad (P_2 L \Psi'(z), v)_{W^{1,2}} = \frac{1}{2\pi} \int_0^{2\pi} \bar{H}_q(z) \cdot v dt.$$

The right hand side of (3.49) is a continuous linear functional on $W^{1,2}(S^1, \mathbb{R}^n)$.

Therefore there exists a unique $\Theta = \Theta(z) \in W^{1,2}(S^1, \mathbb{R}^n)$ such that

$$(3.50) \quad (\Theta(z), v)_{W^{1,2}} = \frac{1}{2\pi} \int_0^{2\pi} \bar{H}_q(z) \cdot v dt.$$

Clearly $\Theta(z) = P_2 L \Psi'(z)$. But the map $z \mapsto \bar{H}_q(z)$, $M \rightarrow L^2(S^1, \mathbb{R}^n)$ is continuous and the map $\bar{H}_q(z) \mapsto \Theta(z)$, $L^2(S^1, \mathbb{R}^n) \rightarrow W^{1,2}(S^1, \mathbb{R}^n)$ is compact and linear. It follows that $P_2 L \Psi'$ is compact.

The final result in this section is a version of the so-called "Deformation Theorem" which is appropriate for our setting. A subset $S \subset E$ will be called invariant if $z(t) \in S$ implies that $z(t + \theta) \in S$ for all $t \in [0, 2\pi]$. A mapping $h : S \rightarrow E$, where S is invariant, will be called equivariant if $h(T_\theta z) = T_\theta h(z)$ for all $\theta \in [0, 2\pi]$ where $T_\theta \zeta(t) = \zeta(t + \theta)$. For $s \in \mathbb{R}$, let

$$A_s \equiv \{z \in M \mid A(z) > s\}.$$

For $c \in \mathbb{R}$, let

$$K_c \equiv \{z \in M \mid A(z) = c \text{ and } A'(z) = (A'(z), \Psi'(z))_{\star} \|\Psi'(z)\|^{-2} \Psi'(z)\}_E$$

i.e. K_c is the set of critical points of $A|_M$ having critical value c .

Proposition 3.51: Let $c, \bar{e} > 0$. Then there is an $\epsilon \in (0, \bar{e})$ and $\eta \in C([0, 1] \times M, M)$

such that

1° $\eta(1, \cdot)$ is equivariant

2° $\eta(1, z) = z$ if $A(z) \notin [c - \bar{e}, c + \bar{e}]$

3° $\|\eta(1, z) - z\| < 1$

4° If $K_c = \emptyset$, $\eta(1, A_{c-\epsilon}) \subset A_{c+\epsilon}$

5° $P^+ \eta(1, z) = \beta^+(z)z^+ + B^+(z)$ where $\beta^+ \in C(M, [1, e])$ and $P_2 B^+$ is compact.

Proof: Most of the above assertions follow from standard arguments and therefore we will be somewhat sketchy below. See e.g. [13-15] for more details. The function η is determined as the solution of an ordinary differential equation of the form

$$(3.52) \quad \frac{d\eta}{dt} = \omega(\eta) \left[A'(\eta) - \frac{\langle A'(\eta), L'(\eta) \rangle_*}{\| \Psi'(\eta) \|^2} \Psi'(\eta) \right]$$

$$\eta(0, z) = z \in E.$$

The scalar function ω is Lipschitz continuous, $0 \leq \omega(z) \leq 1$, $\omega(z) = 1$ if $z \in M$ and $A(z)$ is near c . Note that the argument of Proposition 3.23 shows $\Psi^{-1}(s)$ is a manifold for each s near 1, e.g. $|s - 1| \leq s_0$. The function $\omega(z) = 0$ if $|\Psi(z) - 1| > s_0$. Lastly $\omega(T_\theta z) = \omega(z)$ for all $\theta \in [0, 2\pi]$.

Since the right hand side of (3.52) is Lipschitz continuous and is bounded by 1 (see [14] or [15]), there exists a solution of (3.52) defined for all $t \in \mathbb{R}$ and $z \in E$. Moreover $\|\eta(t, z) - z\| \leq 1$ for $t \in [0, 1]$, i.e. 3° holds. The form of (3.52) implies that $\eta(t, M) = M$ for all $t \in \mathbb{R}$. The properties of ω show that $\eta(1, \cdot)$ satisfies 1°-2°. Proposition 3.27 and a standard argument - see [13]-[15] imply 4°. To prove 5°, note that $P^+ A'(z) = z^+$. Therefore integrating (3.52) yields:

$$\begin{aligned}
 (3.53) \quad P^+ \eta(t, z) &= \left(\exp \int_0^t \omega(\eta(s, z)) ds \right) z^+ - \\
 &- \int_0^t \left(\exp \int_0^\tau \omega(\eta(s, z)) ds \right) \omega(\eta(\tau, z)) (A'(\eta(\tau, z), \Psi'(\eta(\tau, z))) \star \\
 &\quad \|\Psi'(\eta(\tau, z))\|^{-2} P^+ \Psi'(\eta(\tau, z)) d\tau .
 \end{aligned}$$

Thus $P^+ \eta$ has the form stated in 5°. The compactness of $P_2 B^+$ follows via Proposition 3.48 and an argument from [16] since P_2 and P^+ commute.

Remark 3.54: Let $A_s = \{z \in M \mid A(z) < s\}$. If we replace $\omega(z)$ by $-\omega(z)$ in (3.52), the assertions of Proposition 3.51 still hold with 4° replace by $\eta(1, A_{C+\epsilon}) \subset A_{C-\epsilon}$ and 5° by $P^- \eta(1, z) = \beta^-(z) z^- + B^-(z)$ where $\beta^- \in C(M, [e^{-1}, 1])$ and $P_2 B^-$ is compact.

§4. Existence of a Solution

The proof of Theorem 1 will be completed in this section. The solution will be obtained as a critical point of $A|_M$ by a minimax argument. Then a simple regularity argument shows it is a classical solution of (HS). The following two lemmas pave the way for the definition of the critical value c .

Lemma 4.1: Let $M^+ \equiv M \cap E^+$ and set

$$\underline{\alpha} \equiv \inf_{z \in M^+} A(z).$$

Then $\underline{\alpha} > 0$.

Proof: By (2.2) for $z = z^+ \in E^+$, $A(z) = \frac{1}{2} \|z^+\|^2$. Since by Proposition 3.25 M is the boundary of a neighborhood of 0 in E , there is an $r > 0$ such that $\|z\| < r$ implies z is interior to M . In particular for $z \in \partial B_r(0) \cap E^+$, $A(z) > \frac{1}{2} r^2$. Hence $\underline{\alpha} > \frac{1}{2} r^2$.

Next let L^+ be a two dimensional invariant subspace of E^+ . We further require that L^+ be such that there is a constant $a_1 > 0$ satisfying

$$(4.2) \quad \|z^+\|_{L^+}^2 < a_1 \|z\|_{L^+}^2$$

for all $z \in E^0 \oplus E^- \oplus L^+$. To find such an L^+ , let e_1, \dots, e_n denote the usual basis in \mathbb{R}^n . Then we can take

$$E^0 \equiv \text{span}\{(e_j, 0), (0, e_k) \mid 1 \leq j, k \leq n\},$$

$$E^+ \equiv \text{span}\{((j+1)\sin jt e_k - (1 + \frac{1}{j})\cos jt e_k),$$

$$(k+1)\cos lt e_m, (1 + \frac{1}{j})\sin lt e_m \mid 1 \leq k, m \leq n, \text{ and } j, l \in \mathbb{N}\},$$

$$E^- \equiv \text{span}\{((j+1)\sin jt e_k, (1 + \frac{1}{j})\cos jt e_k),$$

$$((l+1)\cos lt e_m, - (1 + \frac{1}{l})\sin lt e_m) \mid 1 \leq k, m \leq n \text{ and } j, l \in \mathbb{N}\},$$

$$\text{and } L^+ = \text{span}\{(\sin t e_1, -\cos t e_1), (\cos t e_1, \sin t e_1)\}.$$

It is easy to verify that (4.2) holds.

Lemma 4.3: If $M^- \equiv M \cap (E^- \oplus E^0 \oplus L^+)$ and

$$\bar{a} \equiv \sup_{z \in M^-} A(z),$$

then $\bar{a} < \infty$.

Proof: By Proposition 3.26, M is bounded in $L^2(S^1, \mathbb{R}^{2n})$. Therefore there is an $M_1 > 0$ such that $\|z\|_{L^2} \leq M_1$ for all $z \in M$. In particular for $z = z^- + z^0 + z^+ \in M^-$, by (4.2) we have

$$(4.4) \quad \|z^+\|_{L^2} \leq a_1 M_1.$$

Since L^+ is finite dimensional, there is a constant $a_2 > 0$ such that $\|z^+\|_{L^2} \leq a_2 \|z^+\|_{L^2}$ for all $z^+ \in L^+$. Hence

$$(4.5) \quad A(z) = \frac{1}{2} (\|z^+\|^2 - \|z^-\|^2) \leq \frac{1}{2} (a_1 a_2 M_1)^2$$

for $z \in M^-$ and $\bar{a} \leq \frac{1}{2} (a_1 a_2 M_1)^2$.

Now the class of sets that will be used to find a critical point of $A|_M$ can be introduced. Let

$\Gamma \equiv \{h \in C(M, M) \mid 1^0 \ h \text{ is equivariant,}$

$2^0 \ h(z) = z \text{ if } A(z) \notin [0, \bar{a} + 1],$

$3^0 \ h(z) \text{ maps bounded sets to bounded sets}$

$4^0 \ P^+h(z) = \beta(z)z^+ + B(z) \text{ where } \beta \in C(M, [1, \beta_0]), \beta_0 = \beta_0(h) > 1,$

and $P_2 B(z)$ is compact}.

A critical value c of $A|_M$ can be produced by taking:

$$(4.6) \quad c \equiv \sup_{h \in \Gamma} \inf_{z \in M^+} A(h(z)).$$

To see this, note first that $\text{id} \in \Gamma$. Hence by Lemma 4.1, $c \geq \bar{a} > 0$. To prove that $c < \infty$, the following intersection theorem which is of independent interest is required.

Theorem 4.7: Let $h \in \Gamma$. Then $h(M^+) \cap M^- \neq \emptyset$.

Proof: We will use a finite dimensional approximation argument. Let E_m^+ and E_m^- be $2m$ dimensional invariant subspaces of E^+, E^- respectively such that if $E_m = E_m^- \oplus E^0 \oplus E_m^+$, $\bigcup_{m \in \mathbb{N}} E_m = E$. Such subspaces can be written down explicitly using the basis for E given following (4.2). Let P_m denote the orthogonal projector of E onto E_m . Set $h_m \equiv P_m h \in C(M^+ \cap E_m, E_m)$. Note that by properties 1^o-2^o of Γ , h_m is equivariant and $h_m(z) = z$ on $E^0 \cap M$. By Proposition 2.2 of [17] (where we take f to be the orthogonal projector of E^+ onto the orthogonal complement of L^+ in E^+ composed with h_m), there is a point $z_m \in M^+ \cap E_m$ such that $h_m(z_m) \in E^- \oplus E^0 \oplus L^+$. We claim (z_m) is a bounded sequence. Otherwise $\|z_m\| \rightarrow \infty$ along a subsequence. But then since $z_m \in E^+$, $A(z_m) = \frac{1}{2} \|z_m\|^2 \rightarrow \infty$. Hence by property 2^o of Γ , $h_m(z_m) = z_m$ for large m . Therefore $z_m \in M^+ \cap (E^- \oplus E^0 \oplus L^+) = M \cap L^+$. Since M is bounded in L^2 and L^+ is finite dimensional, (z_m) must be bounded in E , a contradiction.

Thus (z_m) is a bounded sequence. By property 3^o of Γ , $(h_m(z_m))$ is also bounded. Property 4^o of Γ implies that

$$(4.8) \quad q_m = \beta(z_m)^{-1} (P_2 P^+ h_m(z_m) - P_2 B(z_m))$$

Where $z_m = (p_m, q_m)$. The boundedness of (z_m) and compactness of $P_2 B$ show the second term on the right hand side of (4.8) has a convergent subsequence. The boundedness of $h_m(z_m)$ and the fact that $P^+ h_m(z_m)$ lies in L^+ which is finite dimensional implies the first term on the right hand side of (4.8) also has a convergent subsequence. It follows then from (4.8) that q_m has a convergent subsequence in $W^{1,2}(S^1, \mathbb{R}^n)$. Therefore the same is true for $p_m = Dq_m$ in $L^2(S^1, \mathbb{R}^n)$. Consequently $z_m \rightarrow z \in M^+$ and by the continuity of h , $h_m(z_m) \rightarrow h(z) \in M^-$. The Theorem is proved.

Corollary 4.9: $c \leq \bar{\alpha} < \infty$.

Proof: By Theorem 4.7, $h(M^+) \cap M^- \neq \emptyset$ for any $h \in \Gamma$. Therefore for each $h \in \Gamma$,

$$\inf_{z \in M^+} A(h(z)) \leq \sup_{w \in M} A(w) = \bar{\alpha}$$

via Lemma 4.3.

Now we can prove

Theorem 4.10: c is a critical value of $A|_M$.

Proof: If not, we can invoke Proposition 3.51 with $\bar{\varepsilon} = \frac{1}{2} \min(1, \underline{\alpha})$ obtaining

$\eta(1, \cdot) \in C(M, M)$ and satisfying 1^o-5^o of Proposition 3.51. But 1^o-3^o, 5^o, and our choice of $\bar{\varepsilon}$ imply that $\eta(1, \cdot) \in \Gamma$ as is $\eta(1, h)$ for any $h \in \Gamma$. By 4^o of Proposition 3.51,

$$(4.11) \quad \eta(1, \cdot) : A_{c-\varepsilon} \rightarrow A_{c+\varepsilon}.$$

Choose $h \in \Gamma$ so that

$$(4.12) \quad \inf_{z \in M^+} A(h(z)) > c - \varepsilon.$$

By (4.11),

$$(4.13) \quad \inf_{z \in M^+} A(\eta(1, h(z))) > c + \varepsilon.$$

But since $\eta(1, h) \in \Gamma$, (4.6) shows

$$(4.14) \quad \inf_{z \in M^+} A(\eta(1, h(z))) \leq c,$$

a contradiction. Thus c is a critical value of $A|_M$.

Now finally we can complete the

Proof of Theorem 1: Since c is a critical value of $A|_M$, there is a $\lambda \in \mathbb{R}$ and $z \in M$ such that $A(z) = c$ and $A'(z) - \lambda \Psi'(z) = 0$, i.e.

$$(4.15) \quad \int_0^{2\pi} [(p \cdot \dot{Q}) + (P \cdot \dot{q}) - \frac{\lambda}{2\pi} (\bar{H}_p(z) \cdot P) + (\bar{H}_q(z) \cdot Q)] dt = 0$$

for all $(P, Q) \in E$. Equation (4.15) expresses the fact that z is a weak solution of (2.3). The argument of (3.28)-(3.30) and our lower bound for c show $\lambda > 0$. A simple regularity argument - see the proof of Theorem 3.3 of [18] - shows $z \in C^1(S^1, \mathbb{R}^{2n})$, i.e. z is a classical solution of (2.3). Therefore $\bar{H}(z(t)) \equiv \text{constant}$ so $\Psi(z) = 1$ implies that $z(t) \in \mathcal{D}$. Lastly since $\lambda \neq 0$, making the change of time scale $t \rightarrow \lambda t$ shows z is a $2\pi\lambda$ periodic solution of (HS). The proof is complete.

§5. A Dual Approach

In this section another existence proof will be given for a critical value of $A|_M$. This approach is "dual" to the previous one in the spirit of [19]. The critical value obtained in this section may differ from that given by (4.6).

The new critical value will as in §4 be obtained as a minimax. Let

$$\Lambda = \{g \in C(M, M) \mid g \text{ satisfies properties } 1^0-3^0 \text{ of } \Gamma \\ \text{and } 4^0 \text{ } P^-g = \beta^-(z)z^- + B^-(z) \text{ where } \beta^- \in C(M, [\beta_1, 1]), \\ \beta_1 = \beta_1(g) > 0, \text{ and } P_2B^- \text{ is compact}\}.$$

As in §4, there is an intersection theorem associated with Λ .

Theorem 5.1: If $g \in \Lambda$, $g(M^-) \cap M^+ \neq \emptyset$.

Proof: Set $g_m = P_m g \in C(M^- \cap E_m, E_m)$ where E_m and P_m are as in the proof of Theorem 4.7. By properties 1^0-2^0 of Λ , g_m is equivariant and $g_m(z) = z$ on E^0 . Hence Proposition 2 of [17] can again be invoked - this time with f being the orthogonal projector of $E_m^- \oplus E^0 \oplus L^+$ onto $E_m^- \oplus E^0$ composed with g_m - to obtain $z_m \in M^- \cap E_m$ such that $g_m(z_m) \in E_m^+$. By Proposition 3.26, (z_m) is a bounded sequence in $L^2(S^1, \mathbb{R}^{2n})$. Therefore by (4.2), (z_m^+) is bounded in L^2 and therefore in E since $z_m^+ \in L^+$ which is finite dimensional. Since E^0 is L^2 orthogonal to $E^- \oplus E^+$ via the definition of these spaces, (z_m^0) is bounded in E . We claim (z_m^-) is also bounded in E . If not,

$$A(z_m) = \frac{1}{2} (\|z_m^+\|^2 - \|z_m^-\|^2) \rightarrow -\infty.$$

But then by property 2^0 of Λ , $g_m(z_m) = z_m$ for large m so $z_m \in M^- \cap E^+ = M \cap L^+$. This implies $z_m = z_m^+$ for large m , $z_m^- = 0$, and (z_m^-) is a bounded sequence.

Since (z_m) is a bounded sequence, it possesses a subsequence which converges weakly in E to $z \in E$. By property 4^0 of Λ ,

$$(5.2) \quad P_2 P^- g_m(z_m) = 0 = \beta^-(z_m) q_m^- + P_m P_2 B^-(z_m)$$

and $P_m P_2 B^-(z_m)$ has a convergent subsequence in $W^{1,2}(S^1, \mathbb{R}^n)$. Hence so does q_m^- .

Therefore $p_m^- = D_{qm}^-$ does also in $L^2(S^1, \mathbb{R}^n)$. Since $E^0 \in L^+$ is finite dimensional, it follows that $z_m \rightarrow z$ in E and $z \in M^-$. Since g is continuous, $g_m(z_m) \rightarrow g(z) \in M \cap E^+ = M^+$.

Now define

$$(5.3) \quad \tilde{c} = \inf_{g \in \Lambda} \sup_{w \in g(M^-)} A(w).$$

Theorem 5.4: \tilde{c} is a critical value of $A|_M$ with $\underline{\alpha} \leq \tilde{c} \leq \bar{\alpha}$.

Proof: Since $\text{id} \in \Lambda$, $\tilde{c} \leq \bar{\alpha}$. Moreover by Theorem 5.1, if $g \in \Lambda$, $g(M^-) \cap M^+ \neq \emptyset$.

Therefore

$$\tilde{c} \geq \inf_{M^+} A \equiv \underline{\alpha}.$$

Finally using Remark 3.54, the proof that \tilde{c} is a critical value of $A|_M$ follows the same lines as the proof of Theorem 4.10 and we will omit it.

§6. An a Priori Bound for the Period

Theorem 1 establishes the existence of a periodic solution of (HS) as a critical point of $A|_M$. In this section, in a somewhat more general setting, an a priori bound will be obtained for the period of any periodic solution of (HS) in terms of $A(z)$ and various constants determined from $(H_1)-(H_3)$. Writing (HS) in the form (2.3), the period is $2\pi\lambda$; hence our a priori bound is for λ .

Theorem 6.1: Suppose H satisfies $(H_1)-(H_3)$ and $z \in C^1(S^1, \mathbb{R}^{2n})$ is a solution of (2.3) with $\lambda \neq 0$. Then there are constants $\bar{a} > a > 0$ independent of z such that

$$(6.2) \quad a|A(z)| < |\lambda| < \bar{a}|A(z)|.$$

Proof: Without loss of generality we can assume λ and $A(z)$ are positive. Writing (2.3) as

$$(6.3) \quad \dot{p} = -\lambda H_q(z)$$

$$(6.4) \quad \dot{q} = \lambda H_p(z),$$

equation (6.4) implies

$$(6.5) \quad A(z) = \lambda \int_0^{2\pi} p \cdot H_p(z) dt.$$

Consequently

$$(6.6) \quad A(z) \leq 2\pi\lambda \max_{(\xi, \eta) \in \zeta \mathcal{D}} \xi \cdot H_p(\zeta)$$

and this gives the lower bound for λ in (6.2).

Next from (6.3)

$$\begin{aligned} -\lambda \int_0^{2\pi} |H_q|^2 dt &= \int_0^{2\pi} \dot{p} \cdot H_q dt = - \int_0^{2\pi} p \cdot (H_{qq}\dot{q} + H_{qp}\dot{p}) dt \\ &= -\lambda \int_0^{2\pi} p \cdot (H_{qq}H_p - H_{qp}H_q) dt \end{aligned}$$

or

$$(6.7) \quad 0 = \lambda \int_0^{2\pi} [|H_q|^2 + (p \cdot (H_{qp}H_q - H_{qq}H_p))] dt.$$

Adding b times (6.7) to (6.5) gives

$$(6.8) \quad \Lambda(z) = \lambda \int_0^{2\pi} [p \cdot H_p + b|H_q|^2 + (bp \cdot (H_{qp}H_q - H_{qq}H_p))] dt.$$

By (H_2) , there is a $\gamma > 0$ such that

$$|H_q(0,q)| > \gamma \text{ if } (0,q) \in \mathcal{D}.$$

Therefore there is a $\sigma > 0$ such that

$$(6.9) \quad |H_q(p,q)| > \frac{\gamma}{2} \text{ if } (p,q) \in \mathcal{D} \text{ and } |p| < \sigma.$$

Making σ still smaller if necessary, it can be assumed that

$$(6.10) \quad |p \cdot (H_{qp}H_q - H_{qq}H_p)| < \frac{\gamma^2}{8} \text{ if } (p,q) \in \mathcal{D} \text{ and } |p| < \sigma.$$

Writing (6.8) as

$$(6.11) \quad \frac{\Lambda(z)}{\lambda} \equiv I_1 + I_2$$

where I_1 denotes the integral of the right hand side of (6.8) over

$\{t \in [0, 2\pi] \mid |p(t)| < \sigma\}$ and I_2 denotes the complementary integral, lower bounds will be obtained for I_1, I_2 . By (6.9)-(6.11), if

$$l \equiv \text{meas}\{t \in [0, 2\pi] \mid |p(t)| < \sigma\},$$

then

$$(6.12) \quad I_1 > b\left(\frac{\gamma^2}{4} - \frac{\gamma^2}{8}\right)l = b \frac{\gamma^2}{8} l.$$

To estimate I_2 , let

$$M_1 \equiv \max_{z \in \mathcal{D}} |p \cdot (H_{qp}H_q - H_{qq}H_p)|$$

and

$$\omega(\sigma) \equiv \frac{1}{2M_1} \min_{z \in \mathcal{D}, |p| > \sigma} p \cdot H_p(z).$$

Then

$$(6.13) \quad I_2 > (2\omega(\sigma) - h)M_1(2\pi - l) .$$

Choosing $h = \omega(\sigma)$ and combining (6.11)-(6.13) yields

$$(6.14) \quad \frac{A(z)}{\lambda} > \omega(\sigma) \left(\frac{\gamma^2}{8} l + M_1(2\pi - l) \right) \\ > 2\pi\omega(\sigma) \min\left(\frac{\gamma^2}{8}, M_1\right) = \kappa(\sigma) .$$

Thus the upper bound for λ in (6.2) holds with $\bar{a} = \kappa(\sigma)^{-1}$.

Remark 6.15: The constant \bar{a} in (6.2) depends only on C^1 bounds for H on \mathcal{D} while \bar{a} depends on C^2 bounds for H . To obtain an existence theorem for (HS) when H is merely in C^1 , a better estimate for \bar{a} is needed. Let $W(z) \in \mathbb{R}^n$ such that W is C^1 in a neighborhood of \mathcal{D} . Then as in (6.7) we get

$$(6.16) \quad 0 = \lambda \int_0^{2\pi} [(W(z) \cdot H_q) + (p \cdot (W_p(z)H_q - W_q(z)H_p))] dt .$$

Suppose W satisfies

$$(6.17) \quad W(z) \cdot H_q(z) > \frac{\gamma^2}{2} \text{ if } z = (0, q) \in \mathcal{D} .$$

Arguing as in (6.8)-(6.15) then yields

$$(6.18) \quad \frac{A(z)}{\lambda} > \omega(\sigma) \left(\frac{\gamma^2}{8} l + \bar{M}_1(2\pi - l) \right) \equiv \bar{\kappa}(\sigma)$$

where

$$\bar{M}_1 \equiv \max_{z \in \mathcal{D}} |p \cdot (W_p H_q - W_q H_p)| .$$

Therefore we get an upper bound for λ of the desired type.

The existence of a W as in (6.17) follows from a result of Palais [20]. If E is a real Banach space, $0 \subset E$, and $\phi \in C^1(0, R)$, then $w \in E$ is a pseudogradient vector for ϕ at $z \in 0$ if

$$(6.19) \quad (i) \quad \|w\| \leq 2\|\phi'(z)\|$$

$$(ii) \quad \langle \phi'(z), w \rangle_{E^*, E} \geq \|\phi'(z)\|^2.$$

If $\phi \in C^1(E, \mathbb{R})$, $\tilde{E} = \{z \in E \mid \phi'(z) \neq 0\}$, $w(z)$ is locally Lipschitz continuous, and $w(z)$ is a pseudogradient vector for all $z \in \tilde{E}$, $w(z)$ is called a pseudogradient vector field on \tilde{E} . Palais has proved [20].

Lemma 6.20: If $\phi \in C^1(E, \mathbb{R})$, there exists a pseudogradient vector field for w on \tilde{E} .

Choosing $E \equiv \mathbb{R}^{2n}$ and using $(H_1)-(H_2)$, it is easy to verify there exists such a w in our setting. Moreover by using a smooth partition of unity in the proof of Lemma 6.20 - see e.g. Lemma 1.6 of [15] - it can be assumed that w is smooth. Thus the estimate (6.2) holds even when $\phi \in C^1$.

§7. A More Refined Existence Theorem

The goal of this section is to prove

Theorem 7.1: Suppose H satisfies

$$(H_1) \quad H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$$

and $(H_2)-(H_3)$. Then (HS) has a periodic solution on $\mathcal{D} \equiv H^{-1}(1)$.

Proof: Since H satisfies (H_1) , one can find a sequence of functions $H_m \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ such that H_m satisfies $(H_2)-(H_3)$ and H_m converges to H in the C^1 norm uniformly on compact subsets of \mathbb{R}^{2n} . By Theorem 1, the equation

$$(7.2) \quad \dot{z}_m = \lambda_m J_{H_m} z_m(z_m)$$

has a 2π periodic solution z_m lying on $\mathcal{D}_m \equiv H_m^{-1}(1)$ with λ_m satisfying (6.2) with constants $\underline{a}_m, \bar{a}_m$. Moreover

$$(7.3) \quad \underline{\alpha}_m \equiv \inf_{z \in \mathcal{D}_m} A(z) < A(z_m) < \sup_{z \in \mathcal{D}_m} A(z) \equiv \bar{\alpha}_m.$$

We claim there exist constants $\alpha^* > \alpha_* > 0$ such that

$$(7.4) \quad \alpha_* < \underline{\alpha}_m < \bar{\alpha}_m < \alpha^*$$

for all $m \in \mathbb{N}$ and constants $a^* > a_* > 0$ such that

$$(7.5) \quad a_* A(z_m) < \lambda_m < a^* A(z_m)$$

for all large $m \in \mathbb{N}$. Assuming (7.4)-(7.5) for the moment, it follows that the sequence

(λ_m) is bounded away from 0 and ∞ . Since \mathcal{D}_m is near \mathcal{D} for all large m , the functions z_m are bounded in $L^\infty(S^1, \mathbb{R}^{2n})$. Hence by (7.2), (z_m) is bounded in $C^1(S^1, \mathbb{R}^{2n})$. The Arzela-Ascoli Theorem and (7.2) then imply that (λ_m, z_m) converges in $\mathbb{R} \times C^1(S^1, \mathbb{R}^{2n})$ to (λ, z) satisfying

$$\dot{z} = \lambda J_{H_2}(z).$$

To complete the proof of Theorem 7.1, (7.4)-(7.5) must be verified. For the latter inequalities, the constant \underline{a}_m is determined from (6.6) with H replaced by H_m .

Since $H_m \rightarrow H$ in C^1 uniformly in a neighborhood of \mathcal{D} , an a_* which works for all large $m \in \mathbb{N}$ can be determined. The same reasoning, together with the proofs of Theorem 6.1 and Remark 6.15 supply an a^* independent of m provided that there exists a $W(z)$

satisfying (6.17) with $H = H_m$ but γ and W independent of m . Since a γ exists such that

$$(7.6) \quad |H_q(z)| > \gamma^2 \quad \text{for } z = (0, q) \in \mathcal{D},$$

the convergence of H_m to H implies there is a $k \in \mathbb{N}$ such that

$$(7.7) \quad H_{mq}(z) \cdot H_{lq}(z) > \frac{\gamma^2}{2}$$

for $z = (0, q) \in \mathcal{D}_m$ and for all $m, l > k$. Therefore W can be taken to be $H_{kq}(z)$.

Lastly to check (6.4), first note that by the construction of \bar{H} in §1, it can be assumed that there are constants r_1, r_2 such that $\bar{H}_m(z) > r_1|z|^2 - r_2$ for all $z \in \mathbb{R}^{2n}$ (independently of m). Therefore $z \in M_m \equiv \Psi_m^{-1}(1)$ implies that $\|z\|_{L^2}^2 < 2\pi(1 + r_2)r_1^{-1}$. The proof of Lemma 4.3 then shows how to obtain α^* . To get α_* , we argue indirectly. If there were no such constant, then for each $m \in \mathbb{N}$, there is a $\zeta_m \in M_m^+$ and such that $\|\zeta_m\| \rightarrow 0$ as $m \rightarrow \infty$. Suppose $\zeta_m = (\xi_m, \eta_m)$. Then $\eta_m \rightarrow 0$ in $W^{1,2}(S^1, \mathbb{R}^n)$ and a fortiori $\eta_m \rightarrow 0$ in $L^\infty(S^1, \mathbb{R}^n)$ while $\xi_m \rightarrow 0$ in $L^2(S^1, \mathbb{R}^n)$. Since $V_m(q) = U_m(q) = \rho^{-1}|q|^2$ for small q independently of m via the definition of \bar{H} in §1, $\eta_m \rightarrow 0$ as $m \rightarrow \infty$. By (K_4) of Proposition 1.3,

$$K_m(z) \leq r(1 + |z|)|p|$$

where r is independent of m . Consequently $K_m(\zeta_m) \rightarrow 0$ as $m \rightarrow \infty$. But then

$1 = \Psi_m(\zeta_m) \rightarrow 0$ as $m \rightarrow \infty$, a contradiction. The proof is complete.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The main result of this paper is the following theorem: Let $p, q \in \mathbb{R}^n$, $H = H(p, q) \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ and let $H^{-1}(1)$ be the boundary of a compact neighbor- hood of 0 with $\nabla H \neq 0$ on $H^{-1}(1)$. If further $p \cdot H_p > 0$ on $H^{-1}(1)$ when $p \neq 0$, then the Hamiltonian system of ordinary differential equations $\dot{p} = -H_q(p, q), \quad \dot{q} = H_p(p, q)$ possesses a periodic solution on $H^{-1}(1)$. The proof involves minimax arguments from the calculus of variations.		

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